What is Fundamental in Fundamental Physics?

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Abstract

Metaphysicians as well as philosophers of science often turn to particle physics for a description of the most fundamental entities in our universe. The common assumption is that it readily provides one clear account of what those fundamental building blocks are, how they come together to form more complicated objects, and, conversely, how compound objects can be seen as being composed of those fundamental entities. I argue that the picture is more difficult: fundamentality is commonly held to be a relational notion, explicating an ontological hierarchy between compound and fundamental entities. However, particle physics allows for more than one metaphysically meaningful procedure to decompose a system into parts, fundamental or otherwise. I will identify and interpret two commonly used decomposition procedures for quantum systems and show that they lead to different results for what the parts of a quantum system might be and thus give rise to conflicting conceptions of fundamentality.

On the one hand, there is the tensor product decomposition, which is often used to identify as parts of the system clusters of properties that are statistically independent of each other in the sense that a measurement on one of the clusters does not disturb a measurement on other ones. On the other hand, the direct sum decomposition describes the compound system as a mixture of subsystems which each differ in some of the fundamental properties that characterize quantum systems in particle physics—for example electric charge or colour charge. This decomposition also relates to Wigner's "definition" of elementary particles. These represent two very different ways of decomposing a system into its fundamental parts, with disagreeing results, and from the perspective of particle physics both are, often simultaneously, equally valid. I take this to provide a sense in which, as a result, particle physics on its own is not enough to determine the fundamental ontology of the world. This shows that there are conventional choices involved in finding the fundamental parts of an object which have not yet been widely recognised by either metaphysicians or philosophers of science.

1 Introduction

Metaphysicians as well as philosophers of science often turn to modern particle physics for an account of the most fundamental entities in our universe. Tahko (2018, p. 1) observes that many think "that particle physics aims to describe the fundamental level of reality, which contains the basic building blocks of nature." Oppenheim and Putnam (1958, p. 9) put elementary particles at the very bottom of their mereological hierarchy of material entities. Inman (2017, p. 75) claims that "[t]hough the strong reductive letter of Oppenheim and Putnam's account of the mereological ordering of reality has been largely abandoned [...], many contemporary philosophers are apt to endorse something similar in spirit" and points to Kim (1998, p. 15) who asserts that "[t]he bottom level is usually thought to consist of elementary particles, or whatever our best physics is going to tell us are the basic bits of matter out of which all material things are composed" and that "[t]he ordering relation that generates the hierarchical structure is the mereological (part-whole) relation."

The common assumption is that particle physics readily provides an account of what those fundamental building blocks are and how they come together to form more complicated objects, and, conversely, how compound objects can be seen as being composed of those fundamental entities. Even those who examine mereology of quantum theories in more detail, such as Calosi and Tarozzi (2014), tacitly assume that matters are settled in physics regarding how to decompose a given system into its fundamental constituents in the quantum realm.

I argue that this is mislead: particle physics allows for more than one metaphysically meaningful procedure to decompose a system into parts, fundamental or otherwise. In particular, I will explore two decomposition procedures for quantum systems and show that they lead to different results for what the parts of a system might be, thus giving rise to conflicting conceptions of fundamentality. From this I will conclude that particle physics on itself can not provide an account of the fundamental level of reality, and that more interpretational and metaphysical work needs to be done in order to arrive at such a description.

More concretely, I will argue that there are situations where the formal description of a given system in particle physics can be decomposed according to two very different mathematical procedures: the tensor product decomposition identifies parts of the system with clusters of properties that are statistically independent of each other in the sense that a measurement on one of the clusters does not disturb measurements on other ones. On the other hand, the direct sum decomposition describes the compound system as a mixture of subsystems which each differ in one of the fundamental properties that particle physics predicts quantum systems to have—for example electric charge, or color charge. For example, a simple structure like the model used to describe a hydrogen atom can be viewed as either two statistically independent spin-$\frac{1}{2}$ degrees of freedom (the electron and the proton) or a mixture of a spin-0 system, associated with anti-aligned spins of proton and electron, or a spin-1 system associated with two aligned spins, where the theory does not predict which of the two possibilities in the mixture will actually obtain. That is, the behaviour of a hydrogen atom is both determined by two independent spin-$\frac{1}{2}$ systems as well as a spin-0 and a spin-1 system. These represent two very different ways of decomposing the system into its fundamental parts with disagreeing results, and from the perspective of particle physics both are simultaneously equally "correct." Hence, metaphysicians and philosophers of physics need to specify in more detail which of the conceptions of fundamentality they refer to when claiming that particle physics describes the fundamental level of reality, or an account of how two conflicting notions of fundamentality can coexist is needed.

I will proceed as follows: in Section 2, I will first discuss the metaphysics of fundamentality and extract what appears to lie at the core of various accounts, namely a relational concept of ontological priority. Then, I outline the formal description of the basic objects of inquiry of particle physics, quantum systems. The two sections after that then rehearse the two ways in which such a quantum system could be decomposed from a general perspective, and examine how each of them can be justified as corresponding to a relation characterizing ontological hierarchy. Section 5 will then look at the context of particle physics, where the framework of group representations enables us to see exactly where the two notions of fundamentality will clash. Finally, in Section 6, I will show that the two notions disagree on the fundamental constituents that they ascribe to some systems, and conclude that this is in conflict with the expectation that particle physics settles the question of fundamentality for a naturalistic metaphysics.

2 Fundamentality and quantum systems

The term "fundamental" is used in a wide variety of senses in the metaphysics literature, commonly[1] denoting that something is "basic or primitive" (Tahko 2018, p. 1). Most approaches to fundamentality are relational at their core: as Schaffer (2010, p. 36) observes, "[a]nyone who is interested in what is fundamental [...] must understand some notion of priority." That is, fundamentality is connected to a priority relation that holds between the more and less fundamental entities.

This relation is often taken to be that of *grounding* (see e. g. Cameron 2016; Mehta 2017; Schaffer 2009) although this is not accepted by everyone (e. g. Wilson 2014). Some think that there can be multiple such relations: Bennett (2017), for example, argues for a plurality of "building relations" that apply in different circumstances. And others argue (e. g. Fine (2001) and Wilson (2014)) that fundamentality is a primitive notion not further analysable, though still characterizable, perhaps in terms of other relations. Whatever the details of the account of fundamentality might be, common to most approaches—and the only necessary assumption for my argument—is that they require such a relation. For simplicity I shall refer to this relation as the *priority relation*, giving rise to (or being given by) a notion of *(de-)composition* into fundamental constituents—the reader is welcome to substitute

[1]For a survey of notions of fundamentality in metaphysics and philosophy of physics, see Morganti (2020a,b).

their favourite fundamentality relation, if they so wish.

There are two different ways considered in the literature to define what is fundamental using a given priority relation. On the one hand, we can say that x is fundamental if and only if there is no (other) y that is prior to x—this is sometimes referred to as the *independence* conception of fundamentality (e. g. in Tahko (2018) and Bennett (2017, ch. 5)). On the other hand, one can take x to be fundamental just in case it is a member of a set (called a *minimal basis*) $\mathcal{B}$, which is such that for every other entity $y \notin \mathcal{B}$, there are some $b \in \mathcal{B}$ which are prior to y (and which are the only objects prior to y). How the two definitions are related is a topic of debate,[2] but, again, we just note that both of them employ a priority relation between fundamental and non-fundamental entities.

Applied to particle physics, the independence conception of fundamentality corresponds to the claim that elementary particles are not ontologically secondary to some even more fundamental particles in that they don't have structure which could be used to divide them up even further. The notion that elementary particles form a minimal basis of the material world corresponds to the claim that all of matter is made up of these particles, in other words, that they feature as the "building blocks of reality."

Both of these claims are frequently made in the literature; however, in the following I will argue that one is mislead thinking that particle physics readily provides these notions. I will examine two procedures that relate descriptions of quantum systems to what one could think of as their component parts—one based on the direct sum, '$\oplus$', and the other one applying the tensor product, '$\otimes$'—and will show that each can be given a relevant metaphysical interpretation. In the context of particle physics, which extensively employs the mathematical framework of group representations,it will be shown that these notions of fundamentality disagree on the fundamental parts of some systems.

Before we can continue, however, we need to clarify some technical concepts used in particle physics. The basic theoretical framework which underpins particle physics is quantum theory,[3] which describes the kinematics and dynamics of (quantum) systems. For our purposes, a *quantum system* is described by a state space $\mathcal{H}$, which is a complex, separable Hilbert space, together with an algebra of observables

[2] See e. g. Leuenberger (2020), who argues that whether the two definitions agree on the entities they designate as fundamental depends on other metaphysical commitments.

[3] I shall refer to non-relativistic quantum mechanics and quantum field theory both as quantum theories, and will be more specific if we need to pick either one of them.

$\mathcal{A}$, standardly[4] chosen to consist of a suitable algebraic completion of a set of self-adjoint operators on $\mathcal{H}$—often, especially in non-relativistic quantum mechanics, this will be the full algebra of bounded operators of $\mathcal{H}$, denoted by $\mathcal{B}(\mathcal{H})$. Together, these mathematical objects allow us to calculate expectation values of observables in a given state, which are interpreted to represent the mean outcome of the experiments associated with those observables, as well as transition probabilities, which specify the likelihood of the system to transition from one state into another. As a simple example, the Hilbert space $\mathcal{H} = \mathbb{C}^2$ together with an algebra $\mathcal{A}$ generated by the Pauli-spin-matrices describes a spin-$\frac{1}{2}$ system with spin as its only degree of freedom.[5] This models, for example, the spin of an electron or proton, or an abstract qubit.

Many metaphysical conceptions of what the fundamental entities of the world might be are compatible with this characterization of a "system": object ontologies, for example, can take them to be descriptions of material entities. Alternatively, there are constructions of Hilbert spaces available that allow for interpretations that take facts or propositions as the fundamental constituents of reality,[6] and similarly one can adopt other metaphysical views. Motivated in part by considerations about symmetry groups that we will focus on in Section 5, many take structural realism as their chosen ontology of quantum physics, like French and Ladyman (2003), Kantorovich (2009), Lyre (2004), and McKenzie (2020). Here, I shall use the term *system* to refer to a broad variety of what fundamental entities could be and thereby remain neutral on the debate on the correct ontology of quantum theories. For example, regardless of whether one considers quantum field theories to be about particles or fields, my considerations apply in both cases, *mutatis mutandis.* Again, the reader is welcome to substitute their favourite ontology of quantum theories.

We are now ready to look at the decomposition relations available in quantum physics in general, starting with the tensor product in the next section, and continuing with the direct sum in Section 4. In Section 5 we will then focus on the particular theoretical frameworks used in particle physics, to further motivate the use of the direct sum decomposition as a natural way to define fundamentality in the context of elementary particles, and to prepare the way for Section 6, where we see how the two notions disagree.

[4] For generalizations see Roberts (2018).

[5] Note that in this case the algebra of observables is the full algebra of (bounded) operators on $\mathbb{C}^2$, that is, $\mathcal{A} = \mathcal{B}(\mathcal{H}) = \mathbb{C}^{2\times 2}$.

[6] See e. g. Jauch (1968) for a construction of the formalism starting with a system of propositions.

3 Tensor product decomposition

The tensor product is introduced in first textbooks on quantum mechanics as the standard way to model compound systems, it is one of the central notions in the literature on entanglement,[7] and it is widely used in constructions in quantum field theory as well. I start by reviewing the formal construction, and will then look at one way to justify the tensor product as a metaphysically meaningful priority relation.

3.1 The tensor product

The tensor product can be formed for both Hilbert spaces as well as algebras of observables. One way to obtain the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ is to consider the Hilbert space which, as a vector space, is spanned by vectors of the form $e_i \otimes f_j$, where e_i denote the basis vectors of $\mathcal{H}_1$ and f_j denote the basis vectors of $\mathcal{H}_2$. The inner product on the tensor product space is given by the product of the individual inner products: $\langle u \otimes x, v \otimes y \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} := \langle u, v \rangle_{\mathcal{H}_1} \langle x, y \rangle_{\mathcal{H}_2}$, where $u, v \in \mathcal{H}_1$ and $x, y \in \mathcal{H}_2$. For finite-dimensional complex vector spaces $\mathbb{C}^m$ and $\mathbb{C}^n$, their tensor product is isomorphic to the vector space $\mathbb{C}^{mn}$, so for our example of $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$ we would find the tensor product of two spin-$\frac{1}{2}$ systems to be $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$.

The algebra of observables of this joint system can be constructed as the algebra generated by operators of the form $A_1 \otimes I_2$ and $I_1 \otimes A_2$ where I_1 and I_2 are the identity maps on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, whereas A_1 is from the algebra of observables $\mathcal{A}_1$ and A_2 is taken from the algebra $\mathcal{A}_2$. Conversely, if one constructs the tensor product algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ of two algebras there are natural embeddings of the factor algebras into the tensor product given by $\iota_1 \colon \mathcal{A}_1 \to \mathcal{A}, a \mapsto a \otimes 1_{\mathcal{A}_2}$ and similarly for $\mathcal{A}_2$, so that the choice above for algebras of observables agrees with the standard tensor product of the factor algebras. Furthermore, there is an isomorphism relating $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \cong \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Thus, if the algebra of observables consists of all bounded operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then the tensor product of the algebras of observables will again be all the bounded operators of the tensor product space.

Consider the toy example of the Hilbert space $\mathcal{H} = \mathbb{C}^4$ with the algebra of observables consisting of all bounded operators, i. e. $\mathcal{A} = \mathbb{C}^{4\times 4}$. Then this system

[7] See for a conceptual overview Earman (2015).

could be viewed as modelling two spin-$\frac{1}{2}$ systems, as $\mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$, and the corresponding algebras would be again the full algebras of bounded operators on each $\mathbb{C}^2$ subspace.

From a purely mathematical point of view, given a Hilbert space whose dimension is not a prime number, we can thus find non-trivial factor spaces such that the Hilbert space is the tensor product of those factor spaces. That is, given a system modelled by such a Hilbert space, we can identify the factor spaces with subsystems of that system according to the tensor product decomposition. If the algebra of observables of the compound system is the full algebra of bounded operators, then the algebras of the factor spaces will again be the respective algebras of all bounded operators. If the algebra of the compound system is not the full algebra, then there might not be a suitable tensor product decomposition and the system might must be regarded as fundamental according to this notion of fundamentality.

3.2 Statistical independence

How can we interpret the tensor product decomposition of a given quantum system into component parts? One way to arrive at this construction is the requirement of the subsystems being *statistically independent* from each other.[8] The idea is that parts of a compound system should be suitably independent of each other to properly call them parts. Hence, in looking for the components of a compound system one tries to find clusters of properties (or in the language of quantum mechanics: subalgebras of the algebra of observables) that are statistically independent of each other in the sense that a measurement on one cluster does not disturb the results of measurements on another one.

One of the simplest conditions[9] expressing statistical independence used in the framework of quantum theories is that the observables for each of the subsystems commute: that is, for all $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$ we have that $[A_1, A_2] = 0$, where $\mathcal{A}_1, \mathcal{A}_2$ are the algebras of observables associated with the respective subsystems, embedded in the algebra of the compound system. One can arrive at this requirement in multiple ways. Malament (1996, p. 5, footnote 5) for example shows that two observables commuting is equivalent to the conditional probabilities, con-

[8]This notion of independence should not be confused with metaphysical independence introduced in section 2.

[9]See for an overview of such conditions in the context of quantum theories Summers (2009). The notion of "statistical independence" there is only a special case of what I consider here to be a broader category of possible requirements on subsystems.

ditioned on the measurement of the respectively other observable, being equal to the non-conditional probabilities. To illustrate, consider two observables A, B and the probabilities of the values of these observables being in certain sets $a, b \subseteq \mathbb{R}$, denoted by $P(A \in a)$ and $P(B \in b)$. Consider now the probability $P(B \in b|A \in a)$, i. e. the conditional probability of the value of the observable B being in set b, given one already measured the system with respect to observable A and that value was in a. This can be calculated using the so-called "Lüders rule", and Malament shows that $P(B \in b|A \in a) = P(B \in b)$ is equivalent to the associated operators A and B commuting. That is, if (and only if) the observables are commuting, the predictions of the theory for outcomes of measurements of these observables are independent of each other in the sense that even if one measures A, the resulting probability distribution for B will still remain as if one did not measure A and vice versa.

Unfortunately, the commutativity of the algebras of observables of the component system in the compound system is not enough to guarantee that the compound system is the tensor product of the component systems. One needs stronger conditions on the algebras, and various such conditions are discussed in the literature. I shall only mention the case of one of the strongest of these conditions[10], the so-called *split-property*: two von Neumann-algebras[11] $\mathcal{A}_1, \mathcal{A}_2$, are said to satisfy the split property just in case there exists a Type I factor[12] $\mathcal{F}$ such that $\mathcal{A}_1 \subset \mathcal{F} \subset \mathcal{A}_2'$, where $\mathcal{A}_2'$ denotes the *commutator* of $\mathcal{A}_2$, that is, all observables in $\mathcal{A}$ that commute with all elements of $\mathcal{A}_2$. In the case where both $\mathcal{A}_1$ and $\mathcal{A}_2$ are Type I factor algebras, which is the case in non-relativistic quantum mechanics or when we are dealing with finite-dimensional Hilbert spaces, the split property implies that the smallest algebra containing both $\mathcal{A}_1$ and $\mathcal{A}_2$, commonly denoted as $\mathcal{A}_1 \vee \mathcal{A}_2$ is actually isomorphic to the tensor product of the two (in the sense of von Neumann algebras).[13] That is, in this case the compound system containing only the two component systems is given by the tensor product.

Hence, in the cases we are interested in here, if two von Neumann algebras satisfy statistical independence in form of the split property, then the compound

[10]Discussed in both Summers (2009) and Earman (2015).

[11]A von Neumann-algebra is an algebra of bounded operators on a Hilbert space that is closed in the so-called weak operator topology.

[12]A central result for von Neumann algebras is that each such algebra is isomorphic to a direct integral (a generalization of the direct sum) of so-called factors, which are classified into three Types (I-III). Every $\mathcal{B}(\mathcal{H})$ is a Type I factor. See for more details on this topic e. g. Haag (1996, Section III.2).

[13]See for example theorem 4.1 in Summers (2009, p. 8), together with the fact that for Type I factors the tensor product of von Neumann algebras agrees with the algebraic tensor product used here.

algebra will be the tensor product of the two. This motivates the tensor product as a construction from the requirement of independence of component parts in a compound system. Conversely, if we are given a system and we want to find its parts, then we can look for those which are in this sense independent of each other.

Note, that not every decomposition of a Hilbert space into a tensor product can be given a physical interpretation—when we turn to group representations later, this fact can be formulated more precisely. Additionally, such a decomposition is not unique for a system: there could be several different possible ways how to view a given Hilbert space as the tensor product of factor spaces. This notion is explored in the literature on "virtual subsystems" in quantum information theory, see for example Zanardi (2001).[14] However, it should be noted that this is a different claim from my main argument: whereas virtual subsystems concern the non-uniqueness of decompositions once one has chosen a fixed decomposition procedure (namely the tensor product),[15] I argue here that the tensor product is not the only way to look at the decomposition of a quantum system in principle.

In sum, considering the fact that the tensor product is often used by physicists to model the relation of a compound system to its parts, and since this can be understood conceptually as arising from the statistical independence of the component systems, I take the tensor product to be a priority relation in the sense of Section 2. This gives rise to a meaningful notion of fundamentality in particle physics: a compound system can be broken into its fundamental components by considering tensor product factors as associated with statistically independent systems, and, if it cannot be broken down any further it should be considered *fundamental*. Conversely, fundamental systems can be composed into compound systems by using the tensor product construction.

4 Direct sum decomposition

The other construction that I claim is suitable as a relation giving rise to a notion of fundamentality emerges in cases of systems featuring so-called *superselection rules*. For our purposes,[16] these are systems where the algebra of observables is, in a

[14] That these systems might not be so "virtual" after all is suggested in contexts of quantum optics, see for example Reck et al. (1994).

[15] Of course, the fact that this is a choice is not usually made explicit, which is one of my main criticisms in this paper.

[16] See Earman (2008) for an outline of different ways to define superselection and philosophical considerations thereof.

specific way, a proper subset of the full algebra of all bounded operators on the Hilbert space, i. e. cases in which $\mathcal{A} \subsetneq \mathcal{B}(\mathcal{H})$. In these circumstances, the Hilbert space will decompose into a direct sum of *superselection sectors*, which we can then interpret as component parts prior to the compound system.

In the following I will, as in the previous section, outline the technical construction first and then present a way to interpret and motivate it. Then, in Section 5, I will give another justification using the mathematical framework of group representation theory, which is motivated forcefully by its use in the standard model of particle physics. This will allow us to see the most powerful formulation of my argument, where the two decomposition relations actually disagree on the very same systems.

4.1 The direct sum

The direct sum of two Hilbert spaces $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}$ is given by the Hilbert space whose basis is the disjoint union of the bases of the summands. That is, if e_i are the basis elements of $\mathcal{H}_1$ and f_j are the basis elements of $\mathcal{H}_2$ then $\mathcal{H}$ is spanned by the elements $\{e'_i, f'_j\}$, where the dash denotes that even if some $e_q = f_l$, they are taken to be distinct elements in the direct sum space. The vectors in the sum vector space can then be written as $x = \sum_i c_i e'_i + \sum_j k_j f'_j$, although one usually uses the notation $e_i \oplus f_j$ instead of $e'_i + f'_j$. The inner product of this space is extended accordingly as the sum of the inner products of the component spaces. For finite-dimensional Hilbert spaces $\mathbb{C}^m, \mathbb{C}^n$ this means that $\mathbb{C}^m \oplus \mathbb{C}^n \cong \mathbb{C}^{m+n}$, that is, the dimension of the direct sum is the sum of the dimensions of the summands. The algebras of observables of the sum vector space arise naturally as the direct sums of the algebras of the summand spaces; in the finite-dimensional case the operators are realized as block matrices.

It is important to note that the description of a quantum system is always given by a Hilbert space *and* an algebra of observables, together. Consider again the toy example of $\mathcal{H} = \mathbb{C}^4$. This space can obviously be viewed as the direct sum $\mathbb{C}^4 \cong \mathbb{C}^2 \oplus \mathbb{C}^2$ of $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$. However, even if we consider the two summand spaces $\mathcal{H}_i$ to be equipped with algebras of observables that contain all bounded operators of $\mathbb{C}^2$ (which, of course, are just all two-by-two matrices), then the direct sum of those algebras of observables would still not be the the full algebra of four-by-four matrices. Put differently, if the quantum system is modelled by the Hilbert space $\mathcal{H} = \mathbb{C}^4$ equipped with an algebra of observables that contains all bounded operators of $\mathbb{C}^4$, then the direct sum construction is not available as a decomposition of quantum systems. Hence, this decomposition, similar to the case of the tensor product, is not available in all situations. Again, when we consider all of this in the more concrete setting of particle physics, those difficulties can be addressed.

Note also that the tensor product space $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$ is formally the same vector space as the direct sum of the two spaces, $\mathbb{C}^4 \cong \mathbb{C}^2 \oplus \mathbb{C}^2$. However, the basis vectors of the direct sum are given by the disjoint union of the summand bases, i. e. $\mathbb{C}^4 = \langle e_1, e_2, f_1, f_2 \rangle$ whereas in the tensor product space the basis is $\mathbb{C}^4 = \langle e_1 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_1, e_2 \otimes f_2 \rangle$. Hence, the relationship between the compound space and the component spaces is entirely different in the two cases. Again, it should be noted that the algebras of observables will also differ in the two cases if we use the constructions as composition procedures.

4.2 Superselection and mixed states

How can we interpret this decomposition physically? As mentioned in the introduction to this section, the situations[17] in which this decomposition is available are those in which superselection occurs. In these cases the algebra of observables is missing some operators, namely exactly those that would superpose states from different superselection sectors or transform a state contained in one of the sectors into a state in different sector. The sectors are all subspaces and taken together exhaust the whole Hilbert space.

If the Hilbert space and algebra of observables arise from the direct sum of two Hilbert spaces and associated algebras respectively, then it is easy to see that in general superselection occurs: all observables in this case are of the form $A_1 \oplus A_2$ for $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$ where $\mathcal{A}_1, \mathcal{A}_2$ are the algebras of observables of the two component systems. Since they act separately on the direct sum components of vectors, that is $(A_1 \oplus A_2)(\psi_1 \oplus \psi_2) = (A_1\psi_1) \oplus (A_2\psi_2)$, components from different subspaces cannot be mixed by such observables.

Mixed states are, roughly speaking, states that are not contained in one single sector of such a Hilbert space, as opposed to *pure states*, which are. Hence, if a superselection rule is present, linear combinations of pure states can result in

[17]For the following mathematical characterizations of these notions we shall restrict ourselves to ordinary quantum mechanics, explicitly excluding quantum field theory—the mathematics for the latter case is considerably more difficult, but the conceptual conclusions for our purposes remain largely the same. See Earman (2015, Section 2) for an outline of the differences for superselection rules, and Ruetsche (2004, Section 3) in the case of mixed states. I am also only considering here what Earman (2015) calls "weak superselection", ignoring other senses of superselection.

mixed states, namely in case the pure states are taken from different superselection sectors.[18] For direct sums of models of quantum systems we thus have that they lead to superselection rules, which in turn allow for mixed states of systems to occur.

Systems in mixed states built from states in different sectors suggest an interpretation in terms of being non-fundamental: in a sense, they are "forbidden" combinations of different component states of a system. Although the interpretation of mixed states is a topic of philosophical debate, I will just follow Ruetsche (2004, Sect. 3) and outline some, that are often brought forward in the contexts that are relevant to our discussion here.[19] Systems that allow for superselection are usually interpreted as *mixtures* of component systems and model situations in which either the exact state of the system is not specified,[20] or there are multiple systems in an ensemble, each in a unique state.[21] That is, on one possible interpretation, the mixture models the state of a system before a measurement of a property that is not known to allow for superpositions (such as the charge quantum numbers), but the exact value of that property is unknown. A mixed state on this account represents a system before a measurement that could distinguish which sector that system is in. Another possibility is that mixed states model an ensemble of systems, each in a definite state, but the mixed state describing the whole ensemble.

Applied to group representations, which we will encounter below, Baker and Halvorson (2010, p. 103) interpret the direct sum $X \oplus Y$ of two such representations X and Y as modelling "a mixture of possible charges, so that [the system] may have either charge X or charge Y; the theory doesn't tell us which". That is, the compound system $X \oplus Y$ is interpreted as being decomposable into the two fundamental systems X and Y, and the theory does not explicitly tell which of the two possibilities is realized. Whatever interpretation one favours, on all of them the system itself is compound, and the components of the mixture are structures metaphysically prior to and required for the very definition of the compound system. Thus, one can interpret the compound system as being decomposable into the parts given by this procedure.

[18] Formally, mixed states are defined on the algebra of observables as those states, which can be written as a non-trivial convex combination of other states—but we shall not get into too deep technical details here. For our purposes it suffices to think of mixed states as those arising from superpositions of pure states.

[19] For a more detailed review see Ruetsche (2004). We are here only dealing with cases of what she calls "ordinary quantum mechanics".

[20] That is, not specified for whatever reason—mixed states are sometimes used to model situations in which the specific state is not known, or in which one wants to be ignorant about it.

[21] For a more detailed review of possible interpretations of mixed states, see Ruetsche (ibid., Sect. 3).

One objection to this interpretation of the direct sum as a decomposition relation might be that states in a direct sum *can* be mixed states, but they don't need to be: the system could also be in a pure state contained in one of the superselection sectors, and then there is obviously no interpretation in terms of ensembles or unknown mixtures available. This is of course true, but on the level of the state space and algebra of observables the system *is* conceived as a mixture structurally, and in this sense it is decomposable into more fundamental parts. The particular state the system is in does not matter to our considerations, since we assume that the state space and algebra of observables are given, and there is no point in specifying a state space larger than necessary just to introduce superselection but then ignore all sectors but one.

5 Particle Physics

So far, we have seen two different decompositions that are possible in quantum physics: a system might turn out to be the tensor product of statistically independent subsystems, or it might be a direct sum and represent a mixture of component systems. In some cases, though, the notions are somehow separate and apply in different circumstances. For example, if the algebra of observables is the full algebra of bounded operators, then there cannot be a direct sum decomposition. In contrast, if superselection is present, and thus the direct sum decomposition is possible, then subsystems that are tensor product factors also need to exhibit superselection rules individually. Surely, then, one might respond, in practice it will be clear which one a system is: a compound of independent parts, or a mixture of possibilities.

First, note that this is not an objection to my argument that particle physics does not give rise to a unique notion of fundamentality because it does not feature a unique decomposition relation. The two accounts of decomposition that I present here are conceptually vastly different, so it is still interesting to look at how naturalistic metaphysical conceptions of fundamentality should incorporate this fact of particle physics. But even more, I will show in the following that within the framework of group representation theory, which is widely employed in particle physics, the two notions actually disagree in a very strong sense: I will discuss how a system can be decomposable via both the direct sum as well as the tensor product constructions in the same circumstances, with different constituent systems arising. This, then, is a clear problem for the hope for a simple and unique priority relation for the metaphysics of fundamentality arising directly from particle physics.

In this section we will look at the standard model and see how the direct sum decomposition arises naturally from the ontology of the theory: the first two subsections will introduce the mathematical framework of group representation theory, and subsection 5.3 then discusses how so-called irreducible representations arise naturally as the fundamental entities of particle physics on the direct sum decomposition. In Section 6 I will finally show how the two decompositions disagree and discuss some philosophical consequences of this incompatability.

5.1 The Standard Model

The *standard model of particle physics* comprises a set of theories that together predict several different types of elementary particles,[22] neatly arranged according to a few properties such as mass, spin, electric charge and other "generalized" charges. Each elementary particle is characterized by the values of these properties: the electron, for example, is a spin-$\frac{1}{2}$ particle with an electric charge of -1, weak isospin of $-\frac{1}{2}$, weak hypercharge of -1, a mass of about 9.11×10^{-31}kg and a color charge of $\mathbf{1}$. Mathematically, these quantum numbers correspond to labels of so-called irreducible representations of symmetry groups, so the color charge of the electron $\mathbf{1}$ is not the natural number 1 but the label of the one-dimensional representation of the global color gauge group, and similarly for the other charges.[23] It is in this sense that one can say that the ontology of the standard model is determined by these symmetry groups and their representations.

This account is the basis for a wide-spread "definition" of elementary particles, summarized here by Ne'eman and Sternberg:[24]

> Ever since the fundamental paper of Wigner on the irreducible representations of the Poincaré group, it has been a (perhaps implicit) defi-

[22]The counting depends a bit on the author: David Jeffery Griffiths (2008, p. 50) counts 61 including the then-to-be-discovered Higgs particle; Thomson (2013, Ch. 1) describes the more usual 12 fermions and 5 bosons. The differences are due to whether one considers certain particles as states of one unified particle or as separate particles proper—an issue that certainly deserves more attention, but is out of scope of this paper.

Furthermore, despite using the term "particle" a few times in this section—because it is how those systems in particle physics are standardly denoted—I don't want to commit to any further consequences one might attach to this notion. For a recent overview of the discussions around particles in quantum theories see for example Fraser (2021).

[23]For an alternative account of why the quantum numbers or charges have to be labels of representations see Baker and Halvorson (2010), though no part of my argument depends on the differences.

[24]Ne'eman and Sternberg (1991, p. 327), quoted in Roberts (2011, p. 51).

> nition in physics that an elementary particle 'is' an irreducible representation of the group, G, of 'symmetries of nature'.

The paper that Ne'eman and Sternberg refer to by Wigner (1939) actually does not deal with elementary particles as such, but is focused on a related mathematical problem: the classification of all unitary representations of the Poincaré[25] group. The motivation that Wigner gives for this endeavour is that unitary representations of the spacetime symmetry group can, to a certain extent, replace the equations of motion for quantum systems that are placed in such a relativistic spacetime.[26] Hence, he argues, by enumerating all possible unitary representations of the spacetime symmetry group, one gets a classification of all equations of motion, i. e. all possible dynamics in relativistic spacetime. Wigner proves in the paper that in order to find all possible unitary representations of the Poincaré group, it suffices to find the irreducible unitary representations (or short: *irreps*), as they serve as the building blocks for all other representations. Then he shows that the irreducible representations of the Poincaré group can be classified by only two parameters: $m \in \mathbb{R}, \sigma \in \frac{1}{2}\mathbb{Z}$, which thus can be used as labels for the representations and are physically interpreted as the mass and spin of the particle that is described by the representation.[27]

This is a short characterization of what is often referred to as "Wigner's conception of particles".[28] The other parameters of elementary particles in the standard model, like charge and color-charge, arise in a similar fashion from other symmetry groups, called *internal* symmetries. In the following, I shall describe how one extends Wigner's account to the other properties of elementary particles, giving a brief account of why representations in general, and irreps in particular can characterize those physical systems. We will see that we can take quantum systems in particle physics to be modelled simply by group representations, which provide both the Hilbert space as well as the algebra of observables—instead of specifying $\mathcal{H}$ and $\mathcal{A}$ separately. We will also see that the notions of tensor product and direct sum carry over accordingly.

[25]In the paper, Wigner refers to what we nowadays call the "Poincaré group" as the "inhomogeneous Lorentz group," whereas the "homogeneous" Lorentz group is what is known today simply as the Lorentz group.

[26]The short argument is that the Poincaré group includes the time-translations of a system which, of course, need to agree with the dynamics of the system and vice versa. I shall not deal with the details of this here, but refer to Roberts (2022, Ch. 4).

[27]Note, that not all possible irreps are physically meaningful, cf. Sternberg (1995, p. 147f).

[28]For a more detailed overview see e. g. Kuhlmann (2010, pp. 87ff.).

5.2 Symmetries and representations

Symmetry groups are used in mathematics to describe the structure of objects by collecting transformations that leave that structure invariant. The above-mentioned Poincaré group $\mathcal{P}$ is the symmetry group of Minkowski spacetime, which is the mathematical representation of spacetime according to special relativity, that is, $\mathbb{R}^4$ equipped with the Lorentz metric, often written as $\mathbb{R}^{1,3}$. $\mathcal{P}$ contains the transformations of $\mathbb{R}^{1,3}$ that leave the Lorentz-distance between two spacetime points invariant: it comprises translations in space and time, rotations in space, parity and time reversal, and so-called Lorentz boosts, which describe the transition of a reference frame at rest to one moving at a constant speed. Similarly, the group of unitary operators on a Hilbert space $\mathcal{U}(\mathcal{H})$ contains the operators that preserve the Hilbert space structure (that is, its linear vector space structure and the inner product).[29] Hence, if $\mathcal{H}$ is used to model a quantum system, $\mathcal{U}(\mathcal{H})$ can be viewed as the symmetry group of the quantum system itself.

Consider now a quantum system with spatio-temporal degrees of freedom, that is, a position in spacetime. Those degrees of freedom cannot be arbitrarily implemented in the state space, because they must respect the structure of spacetime itself. That is, we expect the symmetries of spacetime to be reflected in how the spatio-temporal degrees of freedom are implemented in the Hilbert space representation of the quantum system. This leads to the demand that the symmetries of spacetime shall not alter the basic structure of the quantum system under consideration. In other words, we expect the symmetries of spacetime to correspond to symmetries of the quantum system.

Mathematically, this connection between spacetime and quantum symmetries is expressed by way of *unitary group representations*:[30] A group representation is a homomorphism from a group G to the operators on a vector space, and if this vector space is a Hilbert space and all operators in the image of the map are unitary, it is called a unitary representation: $\pi \colon G \to \mathcal{U}(\mathcal{H})$. That is, π is a group-structure preserving map realizing abstract symmetry transformations in G as (not necessarily

[29]In fact not only unitary but also anti-unitary operators preserve the full Hilbert space structure. However, only the unitary operators form a group and if one uses projective representations (see below), the distinction becomes void. For a more thorough explanation see e. g. Roberts (2022, Section 3.4).

[30]Actually, the relevant representations are not the unitary ones but the *projective* representations. This is a technical subtlety that does not bear any conceptual significance but would complicate our treatment significantly. We therefore, as is customary in philosophical treatments of this matter, stick to unitary representations.

distinct) concrete unitary operators on $\mathcal{H}$.

Requiring $\mathcal{H}$ to carry a representation of the Poincaré group thus gives us the means to say formally that a system has a position in spacetime. Conversely, if we have a quantum system whose Hilbert space carries a representation of the Poincaré group, then some of the symmetries of this quantum system can be readily interpreted as being implementations of the spacetime symmetries, connected to how the spatio-temporal degrees of freedom of the system are implemented in $\mathcal{H}$.[31] Thus, we arrive at what Roberts (2022, Chapter 2) calls the *representation view*: a quantum system has spatio-temporal degrees of freedom if and only if its Hilbert space carries unitary representations of the Poincaré group.[32]

We can extend this to other, namely internal, degrees of freedom (or quantum charges):[33] the structure of the spaces in which they take their values should be preserved in the description of a quantum system. For example, color charges take values in a space whose structure group is $SU(3)$, and so we expect a representation of $SU(3)$ on the Hilbert space of any quantum system that is said to have color charges. Just as in the case of spatio-temporal degrees of freedom we thus say that a quantum system has a given degree of freedom if and only if its Hilbert space carries a representation of the symmetry group of the space that this degree of freedom takes its values in. One might call this the *general representation view*.

Carrying a unitary representation has two main consequences for the description of a quantum system: on the one hand, it implements the assumption that a system has certain degrees of freedom, as just discussed. On the other hand, it fixes a property that remains invariant under the symmetry transformations, namely the possible representations the Hilbert space carries—a property, which, obviously, cannot be changed by application of a symmetry transformation on that space.[34] Hence, if one can label all the unitary representations of a group, then one can label the various quantum systems having those degrees of freedom and use these labels as the *quantum numbers* describing the structure of a system—as we have already seen in the case of the Poincaré group, where the labels are identified as mass and

[31]Note, that this argument can also be run in the converse direction: by systematizing the invariance behaviour of physical systems we can infer the symmetries of spacetime, see also Roberts (2022, Ch. 5).

[32]Or the Galilean group, if we are in a non-relativistic spacetime setting; see Castellani (1998) and Lévy-Leblond (1963).

[33]Cf. Roberts (2022, p. 179).

[34]Technically, these invariants can be identified with the eigenvalues of so-called Casimir operators that commute with all other symmetries in a given representation—and in the case of irreps below, will be multiples of the identity.

spin; in the case of internal symmetry groups, we get the other quantum charges, like color charge. This shows how the quantum numbers are not numbers, but in fact labels of group representations.[35]

So by specifying the quantum numbers of a system, one determines a Hilbert space $\mathcal{H}$, together with a set of unitaries $\pi(G) \subseteq \mathcal{U}(\mathcal{H})$ on it. But this is not everything one gets: one can also construct a corresponding algebra of observables from the group representation structure. To be more precise, one can look at the group's Lie algebra,[36] whose generators will give rise to self-adjoint operators via the lie algebra representation,[37] which in turn can be used to generate an algebra of observables. This way, the available observables derive only from the global structure group defining the system's available degrees of freedom. Hence, we take the general representation view to imply that the labels of a symmetry group fully specify the structure of a quantum system: the available states (the Hilbert space) as well as the algebra of observables, which thus consists of all and only of combinations of observables that are determined by the assumed degrees of freedom.

In sum, the ontology of the standard model is captured by the relevant symmetry groups. The Poincaré group describes spatio-temporal degrees of freedom, giving rise to the quantum charges of mass and spin; the internal symmetry groups $U(1)$ and $SU(2)$ together describe the electroweak degrees of freedom and give rise to weak isospin and weak hypercharge; and the internal symmetry group $SU(3)$ describes color-related degrees of freedom characterizing the color quantum number. It is in this way that the "symmetry group of nature" provides a "definition" of the different types of elementary particles.

[35] Note the connection between the charge and the degrees of freedom: saying that the electron has color charge **1** expresses that the electron is in the **1**-representation of $SU(3)$, from which it follows that it does not have any color-related degrees of freedom. Whereas a quark, which is in the **3**-representation of $SU(3)$, does indeed have color degrees of freedom.

[36] The Lie algebra of a Lie group is an algebra associated to the group, representing infinitesimal group transformations near the identity. See Fuchs and Schweigert (1997, Ch. 4) for basic definitions.

[37] A Lie algebra representation is, similarly to a group representation, a structure preserving map into the operators of a Hilbert space. There is a one-to-one correspondence between unitary group representations and Lie algebra representations for so-called simply connected groups. $SU(n)$ are all simply connected, for the Poincaré group and $U(1)$ there are separate arguments why we get operators representing the measurable properties associated to the degrees of freedom of a quantum system from the symmetry transformations.

5.3 Irreducible representations as fundamental systems

First, note that we can define the direct sum of group representations analogously to the case of Hilbert spaces and algebras of observables: given $\pi_1\colon G \to \mathcal{U}(\mathcal{H}_1)$ and $\pi_2\colon G \to \mathcal{U}(\mathcal{H}_2)$ their direct sum is a new representation, $\Pi\colon G \to \mathcal{U}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ where $g \mapsto \pi_1(g) \oplus \pi_2(g)$. Note that in general $\mathcal{U}(\mathcal{H}_1) \oplus \mathcal{U}(\mathcal{H}_2) \subsetneq \mathcal{U}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. We will define the tensor product of group representations similarly in Section 6.

Then, a unitary[38] representation $\pi\colon G \to \mathcal{U}(\mathcal{H})$ is called *irreducible* (or, as already mentioned, an *irrep*) just in case there are no proper non-trivial subspaces of $\mathcal{H}$ that are invariant under the operators in $\pi(G)$, that is, if there are no subspaces that themselves would furnish a representation of G.[39] Irreps have a very important property: as mentioned above, Wigner showed that the irreps of $\mathcal{P}$ function as basic building blocks for all other unitary representations of $\mathcal{P}$. To be more precise, he found that any representation of the Poincaré group is isomorphic to a direct sum of irreps. The Peter-Weyl theorem[40] proves the same for another important class of groups, the so-called compact Lie groups—all the internal symmetry groups that come up in particle physics are of this type. That is, all relevant symmetry groups in particle physics have the property that any of their unitary representations can be decomposed into a direct sum of irreps.

Hence, irreps can be considered fundamental amongst the representations of a given group according to the metaphysician's definition of a minimal basis, as given in section 2: they are the basic building blocks of all other representations of that group, with respect to a priority relation given by the composition based on the direct sum of group representations. It is easy to see that irreps can further be considered fundamental according to the second account of fundamentality discussed above: they are independent (again, in the metaphysician's sense) because they cannot be the direct sums of other representations.

If we combine this formal sense of fundamentality with the general representation view set out in the subsection before, we get the following: every kind of quantum system in particle physics is given by a group representation (irreducible or reducible). By the above-mentioned theorems by Wigner and Peter-Weyl, any such system is decomposable into a direct sum of irreps. Hence, every system in

[38] The definition holds for any group representations, but we restrict our attention to unitary ones.

[39] More precisely: there exists no closed subspace $V \leq \mathcal{H}$ such that $\forall \psi \in V \forall g \in G\colon \pi(g) \cdot \psi \in V$ except for $V = \mathcal{H}$ and $V = \{0\}$.

[40] See e. g. Sternberg (1995, p. 179).

particle physics is either irreducible or decomposable into a direct sum of irreps. On the representation view, those component parts correspond to component systems carrying the specified degrees of freedom. Overall, we can say that the decomposition into a direct sum reflects a decomposition of the physical system into its fundamental parts. Following "Wigner's definition", those fundamental components are the elementary particles.

Note that the resulting notion of decomposition is relative to the choice of the symmetry group G. This might be viewed as an advantage in that it allows for a bespoke notion of fundamentality depending on what properties of systems we are interested in—for example, one can capture the idea that something is not decomposable with respect to certain properties, while still being decomposable with respect to a bigger set of properties. On the other hand, one could see this as a disadvantage, because it does not guarantee, strictly speaking, an ultimate notion of fundamentality—one can always introduce a larger symmetry group, and this will lead to a new set of irreps.

However, I view this more as an expression of the fact that we might in the future discover that systems, which were previously considered fundamental, turn out to actually be compound systems. This was the case with atoms, that were once thought to be what their name suggests—indivisible—but are now considered to be compound systems composed of various "elementary" particles. With respect to a "full" symmetry group, describing—in the words of Ne'eman and Sternberg—all "symmetries of nature", the notion of fundamentality would be ultimate in the sense of including all possible properties that can be used to discern parts of systems.

I will not take a position here on the question of whether such a full group is discoverable. For my purposes it is enough to conclude that the theorems by Peter-Weyl and Wigner provide a decomposition relation given by the direct sum of irreps, that explains the particle ontology of the Standard Model and is distinct from the decomposition relation arising from the tensor product construction.

Putting everything together, it was shown in this section that, based on the representation view, a quantum system in particle physics can be described by a unitary representation of the structure groups of the degrees of freedom that the system is assumed to have, which in turn specifies the state space and algebra of observables of the system. This allows a decomposition of the system into a direct sum of irreps, which thus can be interpreted as the fundamental components of that system, since the irreps cannot be decomposed any further on the direct sum decomposition account.

6 Incompatible Decompositions

In the previous section I showed how the direct sum of group representations arises naturally in the context of the standard model of particle physics. Now we shall also apply the tensor product construction to group representations and then see how these two decomposition procedures can disagree. We start by defining the tensor product for group representations, and then consider the example of addition of angular momenta from elementary quantum mechanics. I will, however, present it slightly differently from how it is usually done in textbooks, to highlight the discrepancy between the two possible decompositions. I will conclude the section with some philosophical considerations following from what I have shown so far, namely that it is not possible for metaphysics (and philosophy of physics) to simply assume that particle physics gives us a unique conception of how everything is made of elementary particles.

Similar to the direct sum of group representations, the tensor product of group representations $\pi_{1,2}\colon G \to \mathcal{B}(\mathcal{H}_{1,2})$ is given by the representation $\Pi\colon G \to \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ such that $\Pi(g) = \pi_1(g) \otimes \pi_2(g)$. This construction agrees with the ordinary tensor product of Hilbert spaces, however, the algebra of observables obtained from the tensor product of the group representations is distinct from the tensor product of the individual algebras of observables. In general, it won't be the case that if $\pi_i(G) = \mathcal{B}(\mathcal{H}_i)$ that the algebra of the compound system will again be the all bounded operators, i. e. $\Pi(G) \neq \mathcal{B}(\mathcal{H})$. This is a crucial difference to the simple tensor product algebras of observables: it allows for both the direct sum and tensor product decompositions to be available in the same circumstances—namely when the representation is reducible, which necessitates that the algebra of observables is not the full $\mathcal{B}(\mathcal{H})$.

I will reuse the example that we have encountered before, which is system that is described by the Hilbert space $\mathbb{C}^4$. Assume, that it carries a reducible representation of $SU(2)$, the group cahracterising the structure of quantum systems with angular momentum.[41] It now turns out that there are two ways this representation could be broken down into parts, given by irreps: either into the tensor product of two spin-$\frac{1}{2}$ systems, or into the direct sum of a spin-0 and a spin-1 system. That is, if we are handed a quantum system, whose Hilbert space is $\mathbb{C}^4$ and whose algebra of observables has a specific form, together with the information that this quantum

[41]This information suffices to fix both the Hilbert space as well as algebra of observables, since there is only one reducible representation of $SU(2)$ on $\mathbb{C}^4$.

system has a $SU(2)$-structured degree of freedom, we have two possible ways to distinguish parts of this system: on the one hand as two statistically independent spin-$\frac{1}{2}$ subsystems, and on the other hand as the mixture of a spin-0 with a spin-1 system. Neither the formalism nor any observable in the algebra can tell which one is the "correct" decomposition.

The connection to the addition of angular momenta[42] becomes clear when considering the total angular momentum of an electron in an atom, which is described as a degree of freedom with structure group $SU(2)$. It has two contributions: the spin of the electron itself, as well as the orbital angular momentum arising from it being bound to the nucleus of the atom. These are two independent degrees of freedom, both described by $SU(2)$ and hence modelling the system as the tensor product of two spin-$\frac{1}{2}$ system is appropriate. However, the total angular momentum of the system is not fully determined: it can either be that of a spin-0 or that of a spin-1 system, depending on whether the two angular momentum vectors are aligned or not. That is, from this point of view we are dealing with a mixed system, the parts of which are a spin-0 and a spin-1 system. Hence, this system, depending on which fundamentality relation one takes, is either composed of two spin-$\frac{1}{2}$ systems or a spin-0 and a spin-1 system. Particle physics considerations alone cannot straight-forwardly give a unique answer to the question of what the fundamental constituents of this system are.

That the two decompositions will disagree in other cases too is easy to see, at least for systems described by finite-dimensional Hilbert spaces: the dimension of a tensor products equals the product of the dimensions of the constituent spaces, whereas the dimension of the direct sum equals the sum of the dimensions of the summands. For some structure groups there even exist formulas to calculate the different decompositions, known generally as the Clebsch-Gordan formulas:[43] consider again the case of $SU(2)$. We already saw that the irreps can be indexed by half-integer numbers, so formally one can write $\frac{1}{2}$ to refer to the spin-$\frac{1}{2}$ representation. Exploiting the fact that we thus can use half-integer numbers to refer to the irreps,

[42] See e. g. David J. Griffiths (1994, Section 4.4.3, pp. 165ff)

[43] As seen before, such a decomposition is possible for *any* compact Lie group. Explicit formulas and calculations exist for several special cases relevant in particle physics, most notably $SU(n)$.

one can formally write for $X, Y \in \frac{1}{2}\mathbb{N}$ the Clebsch-Gordan Formula for $SU(2)$:[44]

$$X \otimes Y \cong \bigoplus_{Z=|X-Y|}^{|X+Y|} Z,$$

where Z increases in steps of 1. That is, the tensor product of irreps X and Y is isomorphic in the sense of group representations to the direct sum of irreps Z, where Z ranges from $|X-Y|$ to $X+Y$ and where X, Y, Z are now taken to be just ordinary numbers from the half-integers. For our example of two spin-$\frac{1}{2}$ systems, or, equivalently, a spin-0 and a spin-1 system, the formula reads: $\frac{1}{2} \otimes \frac{1}{2} \cong 0 \oplus 1$. All representations involved here—X, Y and all of the Z—are irreducible.

It is important to note that the above analysis does not merely say that the mathematical formalism is ambiguous as to how the physical system "actually" breaks down into parts, and a proper physical inspection will show whether we are dealing with a system that is *either* a mixture of its fundamental parts *or* consists of independent fundamental parts. Instead, the theory itself cannot tell us which of the possibilities is "more real" and from the perspective of the theory both of them are equally valid views about the system.

One might feel inclined to object that my argument shows that the same mathematical framework can be used to describe different physical situations: for example both sound waves and electromagnetic waves can be described by the same wave equations. In the same way, one might argue, the two decomposition relations above describe two different physical situations that just happen to be described by the same mathematical framework. However, the situation in particle physics is different from the case of waves: in the latter situation, there are other physical and metaphysical ways to differentiate the systems—for example, by observing that the waves propagate in different materials. In the case of particle physics, however, the group representation (including the Hilbert space as well as the algebra of observables) is supposed to be a complete description of the physical system. Hence,if one takes the theory as it is, there is no additional information about the systems to be gained that could distinguish the cases.

[44]See for this case Baker and Halvorson (2010, p. 103) or more generally Larkoski (2019, Section 3.3).

7 Conclusion

I have argued that, contrary to popular opinion, particle physics does not provide an account of "the fundamental" because one cannot uniquely determine the fundamental parts of the systems described by particle physics. This was shown by discussing two possible ways of decomposing a quantum system, the direct sum and the tensor product. In the context of group representations, which is a widely used framework within particle physics, the contrast is especially stark: the same system is decomposable into different "fundamental" constituents, depending on the decomposition procedure one chooses. This shows that there is an element of convention and choice necessary to determine the basic building blocks of the material world. This runs against the idea that particle physics simply presents us with an account of the fundamental.

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